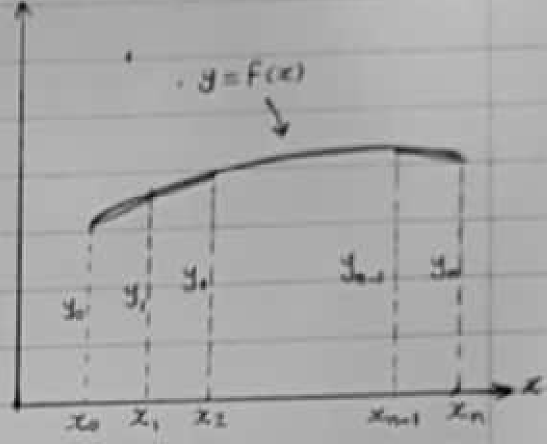
شيماء حجازي

Trapezoidal Method

Consider the integral:

$$I = \int_{x_0}^{x_n} f(x)\, dx$$



Devide the interval $[x_0, x_n]$ to n sub-intervals in such a way that:

$$x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1} = h$$

« equal intervals »

If we know the number of sub-intervals (n) then we can calculate the value of (h) as follows:

$$h = \frac{x_n - x_0}{n}$$

Now, we have n slices, each slide has the shape of trapezoidal.

$$I = I_1 + I_2 + I_3 + \cdots I_n$$

where $I_1, I_2 \ldots I_n$ are the areas of the trapezoidals (slices).

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$$I_1 = \frac{h}{2}(y_0 + y_1)$$

$$I_2 = \frac{h}{2}(y_1 + y_2)$$

$$I_3 = \frac{h}{2}(y_2 + y_3)$$

$$\vdots$$

$$I_n = \frac{h}{2}(y_{n-1} + y_n)$$

$$\therefore I = \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \frac{h}{2}(y_2 + y_3) + \cdots + \frac{h}{2}(y_{n-1} + y_n)$$

$$= \frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-1} + y_n)$$

The error in this formula is of the order h^2

In this method we approximated the function $f(x)$ in each sub-interval as a straight line and this is the source of the error.

To minimise the error we can still use the trapezoidal method but we have to decrease the value of h.

We can think in another method which is the approximation of the function $f(x)$ as a second order formula using three points instead of joining two points by a straight line.

$\therefore$ The first slide will use the three points (x_0, y_0), (x_1, y_1), (x_2, y_2)

The second: (x_2, y_2), (x_3, y_3), (x_4, y_4)

and so on ...

$\therefore$ The no. of sub-intervals should be even number.

Now the problem is : If we have three points, how we can get the function which passes through these three points ?

This problem is called interpolation.

There are many types of interpolation formulae and here we will discuss one of them :

Lagrange's interpolation formula:
for 3 points :

Let $y = f(x)$ be a function which represents or passes through the points $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2).

Since we have three points, then $f(x)$ can be represented by a polynomial of degree 2 :

$$f(x) = a_0 (x - x_1)(x - x_2) + a_1 (x - x_0)(x - x_2)$$

$$+ a_2 (x - x_0)(x - x_1)$$

Where the constants a_0, a_1 and a_2 are to be determined.

if $x = x_0$ then $f(x) = y_0$

$$\therefore \quad y_0 = a_0 (x_0 - x_1)(x_0 - x_2)$$

$$\therefore \quad a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)}$$

if $x = x_1$ then $f(x) = y_1$

$$\therefore \quad a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)}$$

By the same way: $a_2 = \dfrac{y_2}{(x_2 - x_0)(x_2 - x_1)}$

$$\therefore f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

Example :

Find a polynomial of second degree which passes through the points :

x	0	1	2
y	1	2	7

solution :

we have : $x_0 = 0 \quad x_1 = 1 \quad x_2 = 2$

$y_0 = 1 \quad y_1 = 2 \quad y_2 = 7$

substitute there values in the above equation:

$$f(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)} * 1 + \frac{(x-0)(x-2)}{(1-0)(1-2)} * 2 +$$

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$$+ \frac{(x-0)(x-1)}{(2-0)(2-1)} * 7$$

$$\therefore f(x) = \frac{1}{2}(x^2-3x+2) - 2(x^2-2x) + \frac{7}{2}(x^2-x)$$

$$= \frac{1}{2}x^2 - \frac{3}{2}x + 1 - 2x^2 + 4x + \frac{7}{2}x^2 - \frac{7}{2}x$$

$$\therefore f(x) = 2x^2 - x + 1$$

check with the given points.

Home Work:

Find the second degree equation of the following points:

1.

x	0	1	2
y	5	3	3

2.

x	0	1	2
y	4	3	6

3.

x	0	1	2
y	1	-8	-15